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Gravity Force in Capillary Flow Through Open Channels

Silvia A. Wilinski · Aura L. López de Ramos

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Abstract Certain conditions of superficial properties can provoke spontaneous movement of liquids at corners such as liquid filaments. The understanding of this type of fluid dynamics phenomena is very important in many areas, for example, oil recovery or design of gathering systems of condensed fluids with capillary forces. In previous studies, several models were developed for formation and advancing of filaments at horizontal corners neglecting the gravity force. The objective of this investigation was to provide a mathematical tool to estimate the influence of gravity and to establish a clear approach for taking this effect into account or not. The proposed model is developed from a differential equation applied to an open corner for Poiseuille flow. It provides good agreement with experimental values with a maximum deviation of 3.3 %.

Keywords Capillary flow · Filaments · Gravity force · V-shaped channels

1 Introduction

Under certain very specific conditions of surface properties and geometry characteristics, a liquid can move through corners such as filaments by capillary forces [1–3]. Given the small quantities of liquid that flow in a capillary, it can be thought, in some cases, that the effect of the gravity force could be neglected. However, there is an

S. A. Wilinski (✉) · A. L. López de Ramos
Grupo de Investigación Fenómenos de Transporte (GFT), Departamento de Termodinámica y Fenómenos de Transferencia, Universidad Simón Bolívar, Apartado Postal 89000, Caracas 1080A, Venezuela
e-mail: silvia.wilinski@gmail.com

A. L. López de Ramos
e-mail: alopez@usb.ve

important effect of gravity depending on the spatial configuration of a corner and the interfacial properties. For example, it is known that for flow at the corners of a square capillary (vertically oriented) there is an equilibrium configuration that relates the height reached by the liquid columns to interfacial properties and characteristics of the corner [3]. For the case of horizontal corners (such as V-shaped channels), some authors have developed models for estimating the formation and position of filaments. López de Ramos [2,3] obtained an approximate solution of the problem of capillary flow for a corner, and carried out a simplification of Washburn's model [4] for flow inside a cylindrical capillary adapting it to a different geometry (V-shaped corners). In this way, this author obtains an approximate solution similar to the equation of Hagen–Poiseuille. This model relates the flow of the liquid at a corner with the interfacial properties, showing good agreement with respect to experimental data. Later, Romero and Yost [5] and Fuentes [6] improved the model, but only for horizontal corners, neglecting the gravity force since it does not have a contribution in the same direction as the movement of the liquid.

The effect of gravity forces is important in many applications, where the corners can have any space configuration different from the horizontal one. Examples of these cases are in porous media and the design of gathering trays of condensed fluids using capillary forces at corners. The objective of this investigation was to provide a mathematical tool to estimate the influence of gravity in the movement of liquids at corners, and to establish the conditions under which this effect becomes important. For this reason, starting with the existing mathematical models for capillary flow through corners, a mathematical approximation that includes gravity in the balance of forces was developed.

An interesting point on the dynamics of capillary flow through corners occurs when observing in detail a liquid bridge inside a square capillary with a lateral view [7]. In this case, it is possible to appreciate that, even when the principal force of the movement corresponds to the capillary force, the force of gravity modifies the distribution of the liquid, even in horizontal corners, with the result that the thickness of the upper filaments is small compared to that of lower filaments. This case, shown in Fig. 1a, can be simulated by horizontal corners (open up and open down) where the gravity force modifies the value of the thickness of the filament but it does not contribute directly to the movement in the direction of the z-axis. For this reason the dynamics of flow can be modeled with very good results using the approach of the equation of Hagen–Poiseuille adapted by other authors for capillary flow for a V-shaped channel while changing only the initial thickness (Fig. 1b).

This observation is interesting since it allows one to obtain the influence of the gravity. When the channel leans (angle of φ) resulting in rotation of the system of coordinated axes about the x-axis (Fig. 2), the force of gravity can be separated into two components, one along the y-axis and another along the z-axis (both axes rotated by φ) (Fig. 3). In that case, the z-component of the force of gravity affects the movement of the fluid directly in the z-direction, and in this way, it can be considered as the component that contributes to the flow along the channel.

A mathematical model is developed with a differential equation, applied to a fluid that satisfies the conditions of the Hagen–Poiseuille equation and that moves through an open corner. The generic form of this model allows its application fit to different

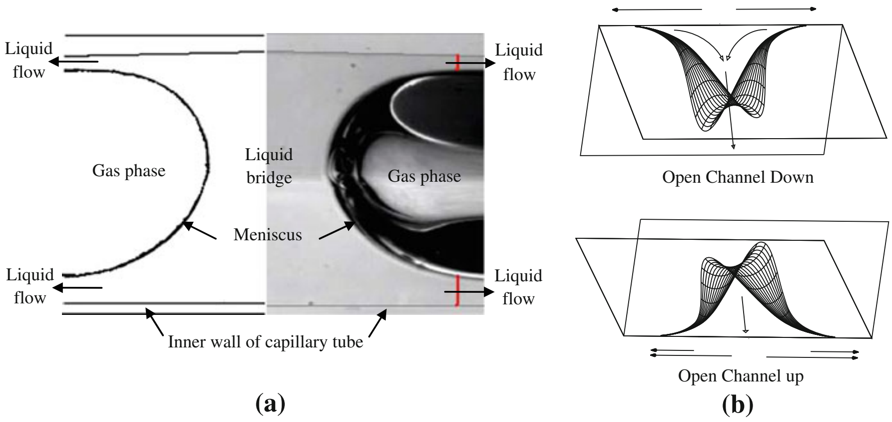


Fig. 1 Analogy of the corners of a capillary horizontal square as V-shaped channels: (a) lateral view of the liquid bridge inside square capillary and (b) model of flow of liquid through corners simulating open channels

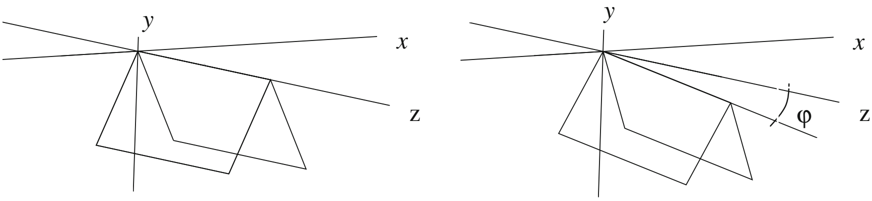


Fig. 2 Angle of inclination of the V-shaped channel

space dispositions of the corner (open up or down, with or without inclination). This equation considers, inside the gradient of pressure, the capillary force and gravity.

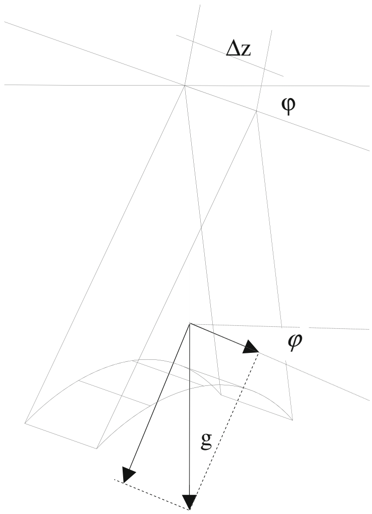
2 Mathematical Model

We consider a V-shaped channel open down, for which there exists a condensation rate such that at $z = 0$ the thickness of the filament is always a maximum ($h = h_{\mathrm{o}}$) (Fig. 4). The flow is assumed to exist in one direction of the channel (from $z = 0$ toward the right or left) in the sense of the glide, and all assumptions for Poiseuille flow are satisfied. The channel can be horizontal or inclined.

Starting from López de Ramos' models [2,3] as approaches of the Washburn's model for cylindrical capillaries [4], and with improvements incorporated by Romero and Yost [5] and Fuentes [6], the volumetric flow was expressed as a function of some properties of the fluid and the interface: dynamic viscosity (μ), filament thickness (h), difference of pressure along the Z-axis ($\partial P/\partial z$), and an adjustment parameter ($\Gamma(\theta, \alpha)$) depending on the contact angle (θ) and geometric angle (α).

$$Q(z,t) = \Gamma(\theta,\alpha)\frac{h^4}{\mu}\frac{\partial P}{\partial z} \tag{1}$$

Fig. 3 Projection of the gravity force for a differential of the inclined filament



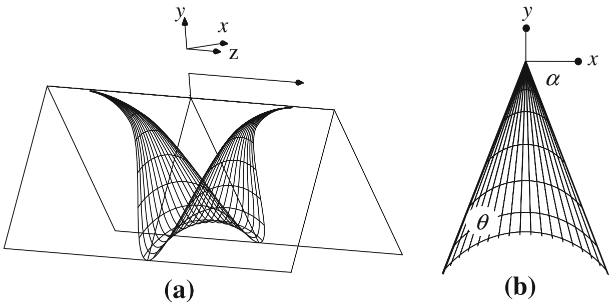


Fig. 4 Formation and advance of filaments in V-shaped channel: (a) lateral view and (b) frontal view

In this mathematical development, the adjustment parameter $\Gamma(\theta, \alpha)$ was used. The adjustment parameter $\Gamma(\theta, \alpha)$ was defined by Romero and Yost [5] (Eqs. 2 and 3), since it provides a good approach to the experimental results [6] and can be applied to different corner angles:

$$\Gamma(\theta, \alpha) = \Gamma(\alpha, \alpha) \left(1 + \cot(\alpha) \frac{\cos(\alpha - \theta) - 1}{\sin(\alpha - \theta)}\right)^3 \left(\frac{\hat{A}(\theta, \alpha)}{\cot(\alpha)}\right)^{1/2} \quad (2)$$

$$\Gamma(\alpha, \alpha) = \left(\frac{1}{6}\right)\left(\frac{\cot^3(\alpha) + 3,4\cot^4(\alpha) + \cot^5(\alpha)}{1 + 3,4\cot(\alpha) + 4\cot^2(\alpha) + 3,4\cot^3(\alpha) + \cot^4(\alpha)}\right) \quad (3)$$

If stationary conditions and an incompressible fluid are assumed, a mass balance applied to a differential volume of the filament of the channel (Appendix A) can be expressed with

$$-\frac{\partial A(z,t)}{\partial t}=\frac{\partial Q(z,t)}{\partial z} \tag{4}$$

The area of the traverse section of flow (A) can be written as a function of the thickness of the filament (h), the geometric angle of the corner (α), and the contact angle (θ),

$$A(z,t)=\hat{A}(\theta,\alpha)h^2 \tag{5}$$

where $\hat{A}(\theta,\alpha)$ is an expression that characterizes the traverse area of flow as a function of the contact angle and the geometric angle,

$$\hat{A}(\theta,\alpha)=\frac{\sin^2(\alpha-\theta)tg(\alpha)-(\alpha-\theta)+\sin(\alpha-\theta)\cos(\alpha-\theta)}{\sin^2(\alpha-\theta)tg^2(\alpha)} \tag{6}$$

When Eqs. 1 and 5 are substituted into the mass balance equation (Eq. 4), the following expression is obtained:

$$\frac{\Gamma(\theta,\alpha)}{\mu}\frac{\partial}{\partial z}\left(h^4\frac{\partial P}{\partial z}\right)=\hat{A}\left(\theta,\alpha\right)\frac{\partial h^2}{\partial t} \tag{7}$$

where an initial condition indicates that the channel is empty at the instant $t = 0$ (Eq. 8); a border condition indicates that the thickness at $z = 0$ remains constant at a value similar to the depth of the channel ($h = h_0$) (Eq. 9). The third condition of the system establishes that the volume of the filament will always be a finite value for any instant of time (Eq. 10).

$$h\left(z,0\right)=0 \tag{8}$$

$$h\left(0,t\right)=h_0 \tag{9}$$

$$\int_0^\infty h^2\left(z,t\right)\partial z<\infty \quad \forall t<\infty \tag{10}$$

Since the volume of liquid is small and is contained in a reduced space, where the area of the contact liquid solid is relatively important, the pressure of Eq. 7 can be written as the sum of several contributions,

$$P=P_{\text{capillary}}+P_{\text{gravity}}+P_{\text{external}} \tag{11}$$

The first contribution corresponds to the capillary pressure; it can be expressed with the Young–Laplace equation, $\Delta P_{\text{capillary}}=\sigma 2H$, which is proportional to the mean curvature of the free surface (H), being the constant of proportionality for interface properties (surface tension) [8]. The surface tension is a property related to spontaneous movement of the liquid under specific conditions. The mean curvature of the free surface (H) represents the arithmetic average of the main curvatures of the surface at a specific point. Considering only two dimensions, there is one in the xy-plane and another in the zy-plane:

$$H = \frac{H_{xy} + H_{yz}}{2} \tag{12}$$

However, the radius of curvature, R_{yz}, goes to infinity ($R_{yz} \to \infty$), and its curvature is equal to zero ($H_{yz} = 1/R_{yz} \to 0$). In this way, for the capillary pressure, only the curvature perpendicular to the z-axis is considered (and $R_{xy} = R_c$) for which the curvature is expressed as

$$H(z) = -\frac{1}{2R_c} = -\frac{\sin(\alpha - \theta)\ tg(\alpha)}{h} = -\frac{w\ (\alpha, \theta)}{h} \tag{13}$$

The second contribution to the pressure in Eq. 11 corresponds to the gravity force. This force can be included in the movement direction (z-axis) in inclined channels for an angle of φ; when projecting the gravity force along the z-axis (rotated) (Fig. 3), then

$$P_{\text{gravity}} = h\rho g \sin(\varphi) \tag{14}$$

The angle of inclination varies between 0° and 90°. When the channel is horizontal, the contribution of the gravity force in the z-direction will be zero; it is important to emphasize that the effect of the gravity force in a horizontal channel is only limited to the variation of the initial thickness of the filament, modifying the movement for this reason, but not for a direct contribution of some force in the direction of the z-axis. The direct contribution of the gravity force to movement through the z-axis will reach its maximum value when the channel is positioned vertically ($\varphi = 90°$). The rest of the possibilities include the repetition of a condition located between these two mentioned angles.

The external pressure is the same at any point on the surface of the filament, annulling its effect on the flow. Then, substituting the expressions from the different contributions to the pressure in the differential equation (Eq. 7) and reorganizing, the following expression is obtained:

$$\frac{D}{h_0^3}\frac{\partial}{\partial z}\left\{h^4\left(\frac{h_0^2}{h^2} + B\right)\frac{\partial h}{\partial z}\right\} = \frac{\partial h^2}{\partial t} \tag{15}$$

where D is defined as a coefficient of capillary diffusion, and B represents the parameter of gravity:

$$D = \frac{\sigma w(\alpha, \theta)\Gamma(\theta, \alpha)}{\mu \hat{A}(\theta, \alpha)} h_0 \tag{16}$$

$$B = \frac{\rho g h_0^2}{\sigma}\frac{\sin(\varphi)}{w(\alpha, \theta)} \tag{17}$$

It can be noted that B includes the Bond number ($\mathrm{Bo} = \rho h_0^2 g/\sigma$) that relates the capillary force with the gravity force. If the value of "Bo" is very small, the effect of gravity can be neglected completely and the differential equation becomes that proposed by Romero and Yost [5].

Consider that the behavior of the filament, in its formation and advance process, will be the same as that expressed by Romero and Yost [5] and Fuentes [6] with the difference of having a different gradient of pressure along the z-axis. The thickness should be zero at some point η_0 (tip of the filament) and should be remain at that value (zero) for further points where the filament does not yet exist. It is possible to carry out a variable change that allows conversion of the partial differential equation into an ordinary differential equation for which an analytical solution for a Taylor series around the point η_0 can be found. To carry out this procedure, a change of variables is used: $h(z, t) = h_o \Phi(\eta)$, with $\eta = z/(1 + B)(Dt)^{1/2}$. The partial differential equation becomes

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{\Phi^4\left(\frac{1}{\Phi^2} + B\right)\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}\right\} = -\frac{(1+B)^2}{2}\frac{\mathrm{d}\Phi^2}{\mathrm{d}\eta} \tag{18}$$

with conditions,

$$\Phi(0) = 1 \tag{19}$$

$$\int_0^\infty \Phi^2(\eta)\,\mathrm{d}\eta < \infty \quad \text{for } \eta < \infty \tag{20}$$

This equation can be solved by means of an expansion of the Taylor series, around η_0:

$$\Phi(\eta) = C_1(\eta - \eta_0) + C_2(\eta - \eta_0)^2 + \mathrm{O}(\eta - \eta_0)^3 \tag{21}$$

When evaluating the differential equation (Eq. 18) for the condition of a finite volume (Appendix B), it is determined that $\mathrm{d}\Phi/\mathrm{d}\eta$ evaluated at η_0 is $-\eta_0(1 + B)^2/2$. If the first term of the series is evaluated at η_0, then it is deduced that $C_1 = -\eta_0(1 + B)^2/2$.

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Phi(\eta) = C_1 + 2C_2(\eta - \eta_0) \tag{22}$$

An expression for C_2 is obtained when substituting the second-order Taylor series into the differential equation (18) and making the coefficients of both sides of the equation equal. The rest of the coefficients can be determined with the same procedure while increasing the level of the series progressively; however, for the results of this study, only the first two terms are considered, since the contribution of the third term is considered negligible. In Table 1 the expressions for the first two coefficients are presented. These coefficients depend on the values of η_0 and the parameters of gravity (tension, density, geometric angle, angle of inclination, contact angle, gravity, and initial thickness of the filament).

Table 1 Coefficients of the Taylor series

Coefficient	Value
C_1	$-\frac{\eta_0(1+B)^2}{2}$
C_2	$\frac{1}{\eta_0} - \frac{(1+B)^2}{2}$

The expression to determine η_0 is obtained when evaluating Eq. 21 at $\eta = 0$. Just as it was already pointed out, this study uses the first two values, since the series will be truncated at the second term ($\eta_0 = 1$). Finally, the equations proposed to model the formation process and advance of liquid filaments in corners considering the effect of the gravity force, allow simulation of the position of the filament front (Eq. 23), the advance velocity (Eq. 24), and the thickness of the filament, for any z at an instant of time t (Eq. 25).

$$z = (1 + B)(Dt)^{1/2} \tag{23}$$

$$V = \frac{(1 + B)}{2}\left(\frac{D}{t}\right)^{1/2} \tag{24}$$

$$h = h_o\left(C_1(\eta - \eta_o) + C_2(\eta - \eta_o)^2 + C_3(\eta - \eta_o)^3\right) \tag{25}$$

3 Experimental Setup

The experimental setup is shown in Fig. 5. A grooved surface was fabricated, carving V-shaped channels on an aluminum plate. The channels have a depth of 1.5×10^{-3}m and an internal angle of the corner of 45°. This grooved surface was washed with water and soap, rinsed with distilled water, and dried in an oven. Later on, it was covered with anti-tarnishes to assure a fulfillment of the relation of Concus and Finn [1] for the formation of liquid filaments spontaneously [9].

Afterward, this surface was placed on a support to change its inclination. The inclination was verified with a conventional meter level (Fig. 5). Several experiments were carried out with the channel: open up, open down, and inclined. The conditions of temperature and pressure were 20 °C and 92.02 kPa for all experiments.

A drop of distilled water was placed at one end of the channel using a syringe with a needle of an appropriate length. The capacity of the syringe body was 1×10^{-3} L which

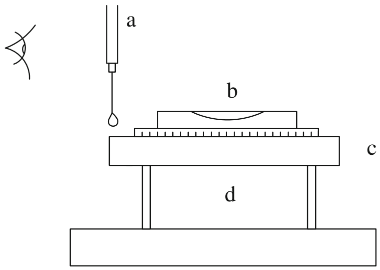


Fig. 5 Experimental setup: (a) syringe, (b) level meter, (c) capillary channels, and (d) support

is an uncertainty of 2×10^{-5} L, and the dimensions of the needle are 0.4 mm × 13 mm. The drop was placed into the groove with finesse, and immediately the liquid filament started to form. This procedure was used not only for the horizontal and inclined faced up channel, but also for the faced down configuration. It is important to point out that the drop can be retained in the groove although the channel was placed faced down, due to the action of capillary forces. The differences in the size of the drops placed at the end of the channel were minimized using the syringe. Registry was made of the time when the filament displaces a distance of 5×10^{-3} m. The position of the front of the filament was determined by direct observation, and the time was measured with a chronometer with a precision of $\pm 1 \times 10^{-3}$ s.

4 Results and Discussion

In Figs. 6, 7, and 8, the experimental values of the distance traveled by the filament versus the square root of the time are presented. In spite of the experimental scatter, it is observed that each of the experimental repetitions describes approximately a straight line. This means that it is possible to model the displacement of the filaments through the corners of the channel with a mathematical relationship of the form of $z = mt^{1/2}$. Probably, the scatter of the experimental values has its origin in human errors of observation, increased by the absence of color of the liquid used and the opacity of the material of the channel. In addition, the small construction differences that can exist among the carved channels, the reaction time from the observer when pressing the chronometer, and lastly, the possible differences in the size of the drop placed inside the channel could also be sources of errors that explain those deviations. The best visual quality was for the inclined channels, which are the results that presented the smallest deviations in the experimental points.

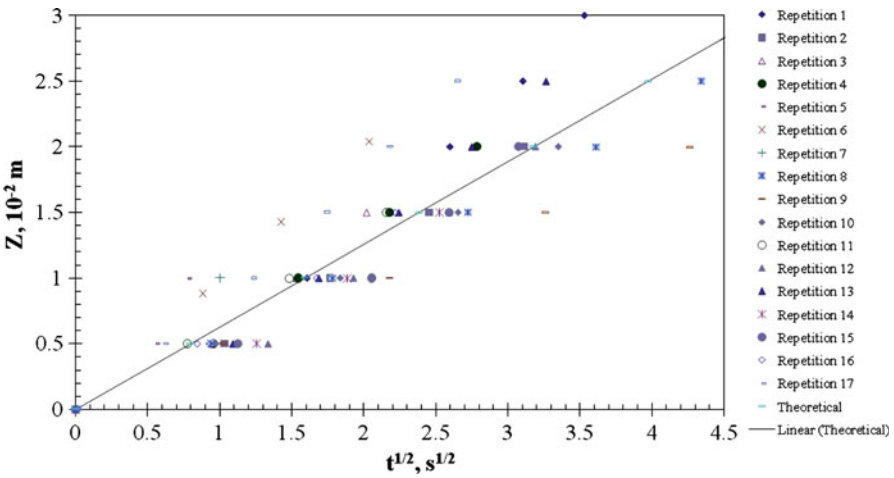


Fig. 6 Results for displacement versus time$^{1/2}$ for horizontal channels (open down)

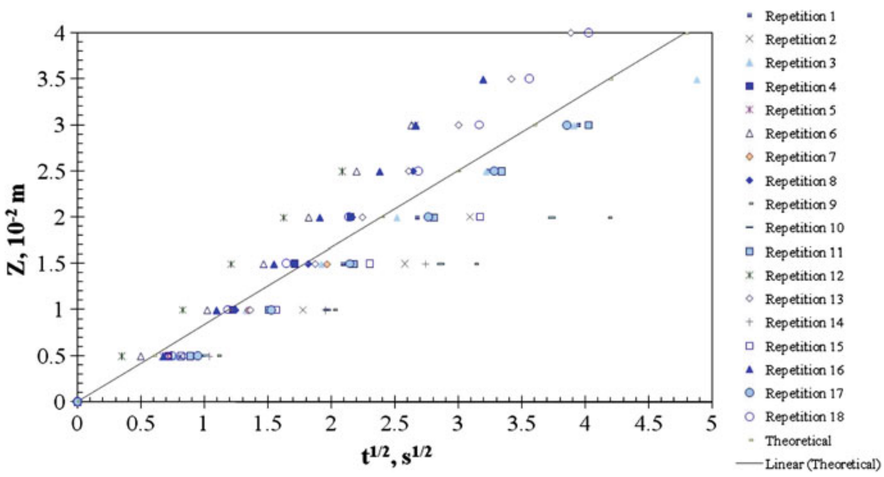


Fig. 7 Results for displacement versus time$^{1/2}$ for horizontal channels (open up)

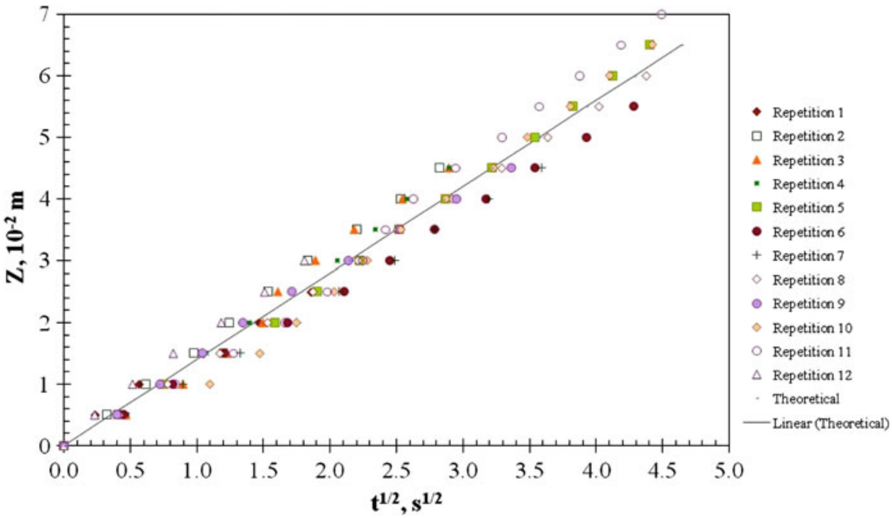


Fig. 8 Results for displacement versus time$^{1/2}$ for inclined channels (10° and open up)

The average experimental values for all cases were compared to each other, using the Student's t-test in order to determine if there were important differences. It was determined statistically (with a level of confidence of 0.99) that the results of the experiments have significant differences to each other. The horizontal cases differ based on the values of the initial thickness of the filament, and they can be simulated with the model obtaining good results. For this reason, it is verified that the gravity force affects the capillary flow for corners depending on the space configuration.

From the values calculated for the slope of the experimental average in each of the cases, it is observed that in the case of 10° inclination, the largest straight line slope was obtained, followed by the horizontal channel open up and lastly the horizontal channel open down. This was expected since the larger is the difference of pressure along the z-axis, the larger will be the displacement obtained. In other words the force of gravity will favor the flow for open corners up and inclined, since the gradient of pressure is larger than in the others.

When the theoretical model is adjusted, the straight lines presented in Figs. 6, 7, and 8 are obtained. It is observed that the distribution of the experimental data is approximately uniform to both sides of the straight line. The properties and parameters used for developing the mathematical model are listed in Table 2.

In Fig. 9 a comparison for the theoretical results of the three setups is shown. It can be noticed that the biggest slope corresponds to the inclined open up, followed by the horizontal open up and the horizontal open down. This result agrees with the experimental observation. As explained before, this is an expected result since the gradient of pressure is bigger in the faced open up than in the faced open down channels.

In Table 3, the mean experimental values are presented with their typical deviations and the theoretical values with their percentage deviations relative to the experimental data. A maximum percentage deviation of 3.30 % was observed between the experi-

Table 2 Properties and parameters of the capillary flow problem

	Properties/parameter	Value	Units
σ	Surface tension	72.76×10^{-3}	$N\cdot m^{-1}$
μ	Dynamic viscosity	1.002×10^{-3}	$Pa\cdot s$
ρ	Density	998	$kg\cdot m^{-3}$
θ	Contact angle[a]	32	$\circ$
α'	Internal angle of the corner[b]	45	$\circ$
g	Acceleration of gravity	9.807	$m\cdot s^{-2}$

[a] Using the equation of Concus and Finn $\alpha'/2+\theta\leq\pi/2$ [1]
[b] $\alpha'=180°-\alpha$

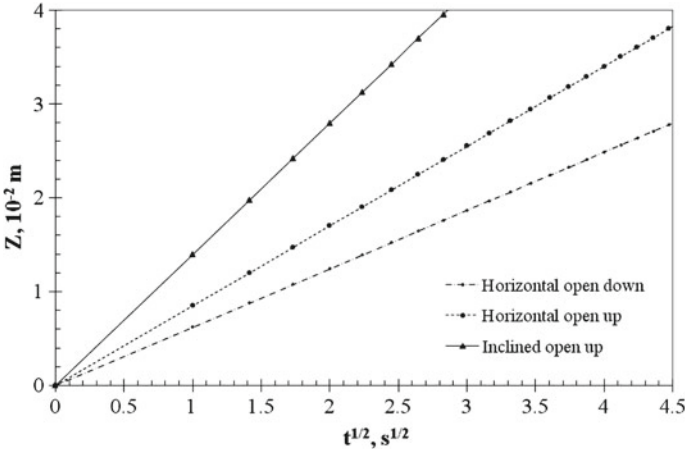


Fig. 9 Comparison between the theoretical results for the setups: horizontal open down, horizontal open up, and inclined open up

Table 3 Results of the different space configurations of the V-shaped channel

Case	Slope, $m \times 10^2$ (experimental) $(\mathrm{m \cdot s^{-1/2}})$	Typical deviation, $m \times 10^2 (\mathrm{m \cdot s^{-1/2}})$	Slope, $m \times 10^2$ (theoretical) $(\mathrm{m \cdot s^{-1/2}})$	% Deviation
Horizontal open down	0.6228	0.1021	0.6213	0.24
Horizontal open up	0.8350	0.2207	0.8508	1.89
Inclined open up	1.4479	0.2229	1.4000	3.30

mental mean and the predicted values by the proposed theoretical model. From this result it can be considered that the model is able to simulate the behavior of the filament in the presence of a gravitational field.

Based on the obtained results, it is accepted that the force of gravity modifies the flow of the liquid in the corners, even when its influence is not decisive in the formation of the liquid filaments. It is also accepted that the proposed mathematical model is able to reproduce the behavior of the liquids inside V-shaped channels.

5 Conclusion

There are some conditions at which it is possible that the surface forces (characterized by surface properties such as contact angle and surface tension) can promote the formation of filaments through corners. These conditions can be used for the movement of liquid from one point to another in a surface with angularities.

Gravity forces can modify the capillary flow at corners according to the space disposition of the corner. Their effect can be increased at higher “Bo” and angles of inclination values. In an inclined channel, bigger displacements of the liquid are obtained due to the force of gravity that contributes directly in the gradient of pressure along the z-axis. The proposed mathematical model, expressed in Eqs. 23–25, incorporates with good results the effect of gravity in the formation and advance of filaments for different filament thicknesses and grades of inclination of the channel. The proposed mathematical model could be used in the design process of collected trays of condensed liquids by capillary forces in different space dispositions of the corner (inclination) since it can be used to estimate with good results the effect of the force of gravity on the flow of the liquid.

Appendix A: Balance of Mass

For the continuity equation, a balance without a chemical reaction on a filament differential can be written as

$$-\frac{\mathrm{d}m}{\mathrm{d}t} = \dot{m}_{\mathrm{s}} - \dot{m}_{\mathrm{e}} \tag{A1}$$

where $\dot{m}_{\mathrm{s}}$ is the flow of exit mass, $\dot{m}_{\mathrm{e}}$ is the flow of entrance mass, and $\mathrm{d}m/\mathrm{d}t$ is the mass accumulated with time.

In terms of the density ρ and of the volumetric flow (Q),

$$-\frac{\mathrm{d}(\rho \cdot V)}{\mathrm{d}t} = \rho_{\mathrm{s}} Q_{\mathrm{s}} - \rho_{\mathrm{e}} Q_{\mathrm{e}} \tag{A2}$$

where the subscripts “e” and “s,” refer, respectively, to the entrance and exit. As the fluid is incompressible ($\rho = \rho_{\mathrm{s}} = \rho_{\mathrm{e}}$):

$$-\frac{\mathrm{d}V}{\mathrm{d}t} = Q_{\mathrm{s}} - Q_{\mathrm{e}} \tag{A3}$$

The volume (V) is similar to the traverse area of the flow (A) for the differential of z, with Δz independent of the time and A a function of the position (z) and of the time (t):

$$-\Delta z \frac{\partial A\,(z,t)}{\partial t} = Q_{\mathrm{s}} - Q_{\mathrm{e}} \tag{A4}$$

$$-\frac{\partial A\,(z,t)}{\partial t} = \frac{Q_{\mathrm{s}} - Q_{\mathrm{e}}}{\Delta z} \tag{A5}$$

It is known by definition that

$$\lim_{\Delta z \to 0} \frac{Q_{\mathrm{s}} - Q_{\mathrm{e}}}{\Delta z} = \frac{\partial Q\,(z,t)}{\partial z} \tag{A6}$$

Finally, the balance of mass is reduced to

$$-\frac{\partial A\,(z,t)}{\partial t} = \frac{\partial Q\,(z,t)}{\partial z} \tag{A7}$$

Appendix B: Condition of Finite Volume

A condition to be completed in the system is that the volume of the filament is finite.

$$\int_0^{\infty} \Phi^2\, \mathrm{d}\eta < \infty \quad \text{when } \eta < \infty \tag{B1}$$

It should be guaranteed that the volume of the filament extrapolates to zero when it comes closer to the tip of the filament:

$$\Phi^2 \frac{\mathrm{d}}{\mathrm{d}\eta} \Phi \to 0 \quad \text{when } \eta \to \eta_0 \tag{B2}$$

The differential equation,

$$\frac{1}{\eta} \frac{\mathrm{d}}{\mathrm{d}\eta} \left\{ \Phi^4 \left(\frac{1}{\Phi^2} + B \right) \frac{\mathrm{d}\Phi}{\mathrm{d}\eta} \right\} = -\frac{(1+B)^2}{2} \frac{\mathrm{d}\Phi^2}{\mathrm{d}\eta} \tag{B3}$$

It is integrated between a point η and η_0;

$$\int_{\eta}^{\eta_0} \frac{\mathrm{d}}{\mathrm{d}\eta}\left\{\Phi^4\left(\frac{1}{\Phi^2}+B\right)\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}+\frac{(1+B)^2\eta}{2}\Phi^2\right\}=\frac{(1+B)^2}{2}\int_{\eta}^{\eta_0}\Phi^2\mathrm{d}\eta \tag{B4}$$

obtaining

$$\Phi^2\left[(1+B\Phi^2)\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}+\frac{(1+B)^2\eta}{2}\right]_{\eta}^{\eta_0}=\frac{(1+B)^2}{2}\int_{\eta}^{\eta_0}\Phi^2\mathrm{d}\eta=\frac{(1+B)^2}{2}\Phi^2 f(n) \tag{B5}$$

where $f(n)$ is a function that forces $\Phi(n)$ to be monotonic and fulfills the condition $f(n)\to 0$ when $\eta\to\eta_\mathrm{o}$

$$(1+B\Phi^2)\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}+\frac{(1+B)^2\eta}{2}-\frac{(1+B)^2}{2}f(n)=0 \tag{B6}$$

where

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}=-\frac{(1+B)^2\eta_0}{2}\quad \text{when } \eta\to\eta_\mathrm{o} \tag{B7}$$

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